Unit 13 – Related Rates

But first ..... Implicit Differentiation

<u>Objective</u> – You will find the derivative of a function using implicit diff.

Used to find the derivative when we <u>cannot</u> explicitly solve for y.

<u>To evaluate</u>:

1) Differentiate each term (take the derivative)
2) Add $\frac{dy}{dx}$ to terms with y.
3) Solve for $\frac{dy}{dx}$

<u>Examples</u>: 1) $x^2 + y^2 = 25$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{2y\frac{dy}{dx}}{2y} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = \boxed{\frac{-x}{y}}$$

2) $3x^2 - 4y^2 = 7$

$$6x - 8y\frac{dy}{dx} = 0$$

$$\frac{-8y\frac{dy}{dx}}{-8y} = \frac{-6x}{-8y}$$

$$\frac{dy}{dx} = \boxed{\frac{3x}{4}}$$

3) $x^3 - y^3 = 6xy$

$f = 6x \quad g = y$

$f' = 6 \quad g' = \frac{dy}{dx}$

$$3x^2 - 3y^2\frac{dy}{dx} = y(6) + 6x\frac{dy}{dx}$$

$$3x^2 - 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$3x^2 - 6y = 6x\frac{dy}{dx} + 3y^2\frac{dy}{dx}$$

$$\frac{3x^2 - 6y}{(6x+3y^2)} = \frac{(6x+3y^2)\frac{dy}{dx}}{(6x+3y^2)}$$

$$\frac{dy}{dx} = \frac{3x^2 - 6y}{6x+3y^2}$$

$$\frac{dy}{dx} = \frac{\cancel{3}(x^2-2y)}{\cancel{3}(2x+y^2)}$$

$$\boxed{\frac{dy}{dx} = \frac{x^2-2y}{2x+y^2}}$$

4) $\dfrac{x^3}{1} = \dfrac{x-y}{x+y}$ Quotient Rule ? UGH!!

Cross Multiply ? ☺

$x^4 + x^3y = x - y$

$4x^3 + y(3x^2) + x^3\dfrac{dy}{dx}$ $\quad f = x^3 \quad f' = 3x^2 \quad g = y \quad g' = \dfrac{dy}{dx} \quad gf' + fg'$

$= 1 - \dfrac{dy}{dx}$

$4x^3 + 3x^2y + x^3\dfrac{dy}{dx} = 1 - \dfrac{dy}{dx}$

$4x^3 + 3x^2y - 1 = -x^3\dfrac{dy}{dx} - \dfrac{dy}{dx}$

$\dfrac{4x^3 + 3x^2y - 1}{(-x^3-1)} = \dfrac{(-x^3-1)\dfrac{dy}{dx}}{(-x^3-1)}$

$$\boxed{\dfrac{dy}{dx} = \dfrac{4x^3 + 3x^2y - 1}{-x^3 - 1}}$$

Homework: 1) $x^2 + y^2 = 100$

2) $5x^2 - 4y^2 = 2xy$

3) $x^3 + xy - 2x = 1$

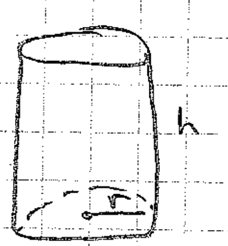
Unit 13 Day #2 – Instantaneous Rates of Change

Objective – You will solve related rates that involve circles, cones, cylinders or spheres.

Instantaneous Rates of Change

1) Sketch and define variables
2) Symbolize known and unknown quantities
3) Write the equation
4) Differentiate
5) Substitute and Solve.

Example #1 Water is flowing into a vertical cylindrical tank of radius 12ft. at the rate of 16 ft^3/min. How fast is the surface of the water rising?



Find: $\frac{dh}{dt}$

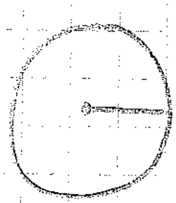
When: $r = 12$ ft

Given: $\frac{dv}{dt} = 16 \, ft^3/min$

$$V = \pi r^2 h$$
$$V = \pi (12)^2 h$$
$$V = 144\pi h$$
$$\frac{dV}{dt} = 144\pi \frac{dh}{dt}$$
$$\frac{16}{144\pi} = \frac{144\pi \frac{dh}{dt}}{144\pi}$$

$$\boxed{\frac{dh}{dt} = \frac{1}{9\pi} \text{ ft/min}}$$

2) Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at the rate of 2 ft/sec. How fast is the area of the spill increasing when the radius of the spill is 60ft



Find: dA/dt

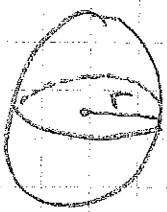
When: $r = 60$ ft

Given: $\frac{dr}{dt} = 2$ ft/sec

$$A = \pi r^2$$
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$
$$\frac{dA}{dt} = 2\pi (60)(2)$$

$$\boxed{\frac{dA}{dt} = 240\pi \text{ ft}^2/\text{sec}}$$

3) A snowball is melting at a rate of 2 ft³/hr. If it remains spherical, at what rate is the radius changing when the radius of the snowball is 1.3 ft.



$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Find: $\frac{dr}{dt}$

When: $r = 1.3$ ft

Given: $\frac{dV}{dt} = -2 \text{ ft}^3/\text{hr.}$

$$-2 = 4\pi(1.3)^2 \frac{dr}{dt}$$

$$\frac{-2}{6.76\pi} = \frac{6.76\pi \frac{dr}{dt}}{6.76\pi}$$

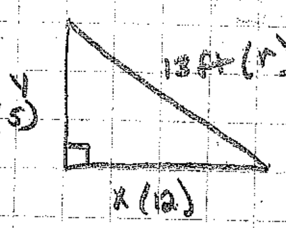
$$\frac{dr}{dt} = \frac{-2}{6.76\pi}$$

$$\boxed{\frac{dr}{dt} = -.0941745225 \text{ ft/hr}}$$

HW: None !! :)

Unit 13 Day #3 – More Related Rates

Example 1) A 13 ft ladder leaning against a wall slips so that its base moves away from the wall at a rate of 2 ft/sec. How fast will the top of the ladder be moving down the wall when the base is 12 ft from the wall?



Find: $\frac{dy}{dt}$

When: $x = 12$ ft

Given: $\frac{dx}{dt} = 2$ ft/sec.

$$x^2 + y^2 = r^2$$

$$x^2 + y^2 = 13^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = 0$$

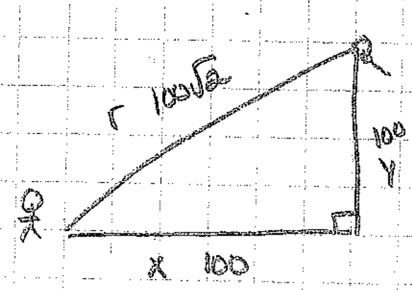
$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$12(2) + 5\left(\frac{dy}{dt}\right) = 0$$

$$5\frac{dy}{dt} = -24$$

$$\boxed{\frac{dy}{dt} = \frac{-24}{5}\text{ ft/sec}}$$

2) A balloon rises vertically at the rate of 10 ft/sec. A person on the ground 100 ft away watches the balloon ascend. At what rate is the distance between the observer and the balloon changing when the balloon is 100 ft off the ground?



Find: $\frac{dr}{dt}$

When: $y = 100$

Given: $\frac{dy}{dt} = 10$ ft/sec.

$$x^2 + y^2 = r^2$$

$$100^2 + y^2 = r^2$$

$$2y\frac{dy}{dt} = 2r\frac{dr}{dt}$$

$$2(100)(10) = 2(100\sqrt{2})\frac{dr}{dt}$$

$$\frac{2000}{200\sqrt{2}} = \frac{200\sqrt{2}\frac{dr}{dt}}{200\sqrt{2}}$$

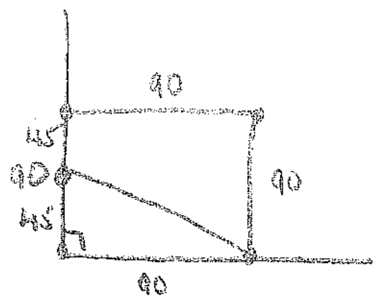
$$\frac{10}{\sqrt{2}} = \frac{dr}{dt}$$

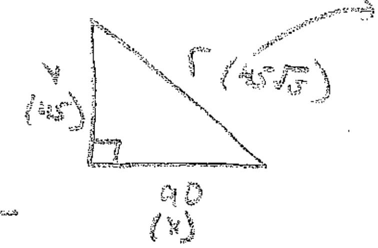
$$\frac{dr}{dt} = \frac{10}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$$

$$\frac{dr}{dt} = \frac{10\sqrt{2}}{2}$$

$$\boxed{\frac{dr}{dt} = 5\sqrt{2}\text{ ft/sec}}$$

3) Each side of a square baseball diamond is 90 ft long. If a ball is hit down the 3rd base line at a speed of 100 ft/sec, how fast is the distance from the ball to first base changing when the ball is halfway to third base?





$r^2 = 45^2 + 90^2$

$r^2 = 2025 + 8100$

$\sqrt{r^2} = \sqrt{10,125}$

$r = \sqrt{25}\sqrt{405}$

$r = 5\sqrt{405}$

$= 5 \cdot \sqrt{9}\sqrt{45}$

$= 5 \cdot 3\sqrt{9}\sqrt{5}$

$= 5 \cdot 3 \cdot 3\sqrt{5}$

$r = 45\sqrt{5}$

Find: $\frac{dr}{dt}$

When: $y = 45$

Given: $\frac{dy}{dt} = 100$ ft/sec.

$x^2 + y^2 = r^2$

$90^2 + y^2 = r^2$

$2y\frac{dy}{dt} = 2r\frac{dr}{dt}$

$2(45)(100) = 2(45\sqrt{5})\frac{dr}{dt}$

$\frac{9000}{90\sqrt{5}} = \frac{90\sqrt{5}\frac{dr}{dt}}{90\sqrt{5}}$

$\frac{dr}{dt} = \frac{100}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$

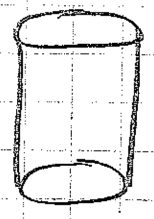
$\frac{dr}{dt} = \frac{100\sqrt{5}}{5}$

$\boxed{\frac{dr}{dt} = 20\sqrt{5} \text{ ft/sec}}$

HW- WS -

Unit 13 Day #4 - Related Rates -

1. A cylindrical tank with a radius of 5 ft and a height of 20 ft is filled with a liquid. A hole is punched in the bottom. At that moment the chemical drains out of the tank at the rate of 2 cubic feet per minute. At what rate is the height of the liquid in the tank changing?



$r = 5ft$

Find: $\frac{dh}{dt}$

When:

Given: $\frac{dV}{dt} = -2 ft^3/min$

$V = \pi r^2 h$

$V = \pi (5)^2 h$

$V = 25\pi h$

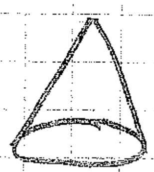
$\frac{dV}{dt} = 25\pi \frac{dh}{dt}$

$\frac{-2}{25\pi} = \frac{25\pi \frac{dh}{dt}}{25\pi}$

$\frac{dh}{dt} = \frac{-2}{25\pi}$ ft/min

$\boxed{\frac{dh}{dt} = -.0254647909}$

2. Sand falling at a rate of 3 cubic feet per minute forms a conical pile whose radius always equals twice its height. Find the rate at which the height is changing at the instant the height is 10 feet.



$r = 2h$

Find: $\frac{dh}{dt}$

When: $h = 10$

$r = 20$

Given: $\frac{dV}{dt} = 3 ft^3/min$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi (2h)^2 h$

$V = \frac{1}{3}\pi 4h^3$

$V = \frac{4}{3}\pi h^3$

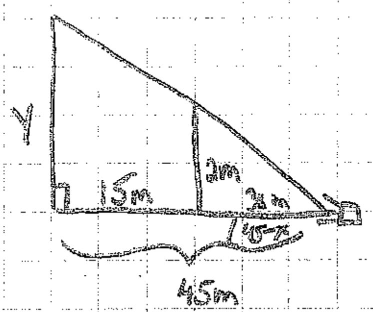
$\frac{dV}{dt} = 4\pi h^2 \frac{dh}{dt}$

$3 = 4\pi (10)^2 \frac{dh}{dt}$

$\frac{3}{400\pi} = \frac{400\pi \frac{dh}{dt}}{400\pi}$

$\boxed{\frac{dh}{dt} = .002387324 \text{ ft/min}}$

Unit 13 - Related Rates - Similar Triangles.



Find: $\frac{dy}{dx}$

when: $x=15$

Given: $\frac{dx}{dy} = -6$ m/sec

$$\frac{y}{45} = \frac{2}{45-x}$$

$$\frac{y(45-x)}{(45-x)} = \frac{90}{(45-x)}$$

$$y = \frac{90}{45-x} \qquad f = 90 \quad f' = 0 \qquad g = 45-x \quad g' = -1$$

$$\frac{dy}{dx} = \frac{(45-x)\cdot(0) - 90(-1)}{(45-x)^2} \frac{dx}{dy}$$

$$\frac{dy}{dx} = \frac{-90}{(45-x)^2} \frac{dx}{dy}$$

$$\frac{dy}{dx} = \frac{-90}{(45-15)^2} \cdot (-6) = \frac{-90}{900} \cdot (-6)$$

$$\boxed{\frac{dy}{dx} = \frac{-3}{5} \text{ m/s}}$$